Chapter 1

Electrostatics

1.1 Coulomb's law

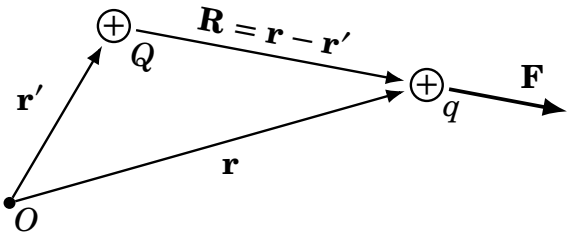


Figure 1.1: Coulomb's law. The source charge Q sits at $\mathbf{r}'$ and the charge q sits at $\mathbf{r}$, which feels the force $\mathbf{F}$ due to Q.

$$\mathbf{F} = k_e \frac{Q\,q}{|\mathbf{R}|^2}\hat{\mathbf{R}}\,, \tag{1.1}$$

where

$$k_e = \frac{1}{4\pi\epsilon_0} \approx 9\times 10^9\ \mathrm{N\cdot m^2/C^2} \tag{1.2}$$

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' \tag{1.3}$$

$$\hat{\mathbf{R}} = \frac{\mathbf{R}}{|\mathbf{R}|}\,. \tag{1.4}$$

1.2 The electric field

Instead of having a Coulomb force between two charges Q and q, we can have a different interpretation: The charge Q does not interact with charge q directly. Instead, the charge Q creates an *electric field* $\mathbf{E}(\mathbf{r})$ around it, and the charge q at position $\mathbf{r}$ feels a force from the electric field $\mathbf{E}(\mathbf{r})$ locally at $\mathbf{r}$.

The electric field $\mathbf{E}$ is a *vector field*. We may think of a vector field as: "at every point in the space $\mathbf{r}$, there is a vector $\mathbf{V}(\mathbf{r})$ (a little arrow) attached to that point $\mathbf{r}$." Because of that, a vector field $\mathbf{V}(\mathbf{r})$ is a function of location, i.e., space coordinate, $\mathbf{r} = (x, y, z)$.

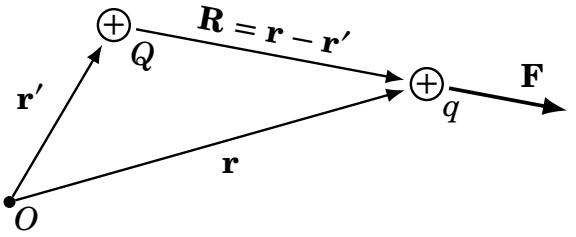


Figure 1.2: Same as Figure 1.1

To make our two-charge example more explicit, we can refer to Figure 1.2. Here, charge Q creates an electric field:

$$\mathbf{E}(\mathbf{r}) = k_e \frac{Q}{|\mathbf{R}|^2}\hat{\mathbf{R}} \tag{1.5}$$

$$= k_e \frac{Q}{|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\,, \tag{1.6}$$

and change q feels the force

$$\mathbf{F}(\mathbf{r}) = q\mathbf{E}(\mathbf{r}) \tag{1.7}$$

$$= k_e \frac{Qq}{|\mathbf{R}|^2}\hat{\mathbf{R}}\,. \tag{1.8}$$

Quantitatively, this is the same as the Coulomb force in the static case, and it seems the $\mathbf{E}$ field is just a mathematical tool to facilitate calculation. However, this is in fact a deep breakthrough in theory, which later on leads to the prediction of the existence of electromagnetic waves.

1.3 Gauss's law

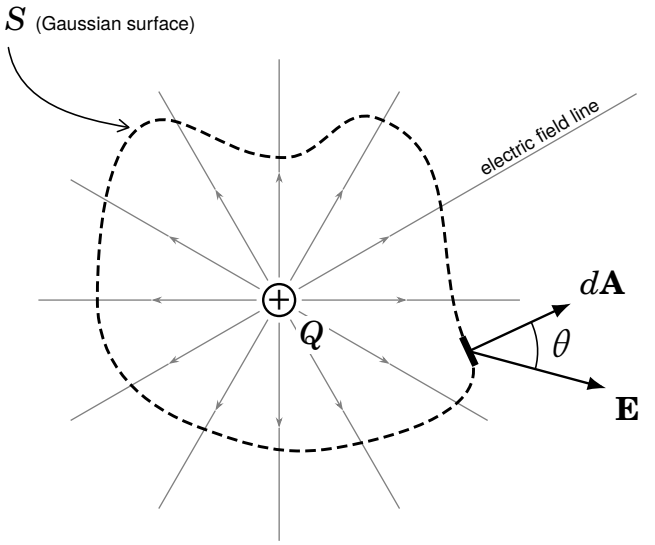


Figure 1.3: A Gaussian surface S of arbitrary shape enclosing the charge Q. At each point of S, the electric field $\mathbf{E}$ meets the area element $d\mathbf{A}$ at an angle θ that varies over the surface. Even so, every electric field line crosses S, so the total flux through S is Q/ϵ_0.

Gauss's law The total electric flux Φ_E through any *closed* surface S, known as a Gaussian surface, equals the total charge Q_{enc} enclosed by S divided by ϵ_0:

$$\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\epsilon_0} \,. \tag{1.9}$$

- The surface integral $\oint_S$ means "add up over the whole closed surface". On each small patch, $d\mathbf{A}$ points outward, and $\mathbf{E} \cdot d\mathbf{A} = E\,dA\cos\theta$ counts only the part of $\mathbf{E}$ that passes through the patch.
- Only the charge *inside* S counts. A charge outside sends as many

field lines in through S as out, so the net flux due to an outside charge is zero.

Since Gauss's law holds for a closed Gaussian surface of any shape, a clever choice of the surface is the key to simplifying the problem.

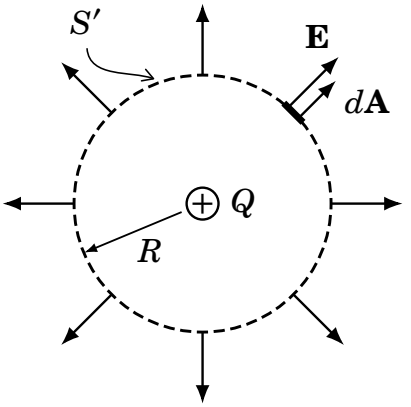


Figure 1.4: A spherical Gaussian surface S' of radius R centered on the charge Q. Everywhere on S', $\mathbf{E}$ is parallel to $d\mathbf{A}$ and has the same magnitude.

$$\Phi_E = \frac{Q_{\text{enc}}}{\epsilon_0} \tag{1.10}$$

$$\Rightarrow EA = E(4\pi R^2) = \frac{Q_{\text{enc}}}{\epsilon_0} \tag{1.11}$$

$$\Rightarrow E = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{R^2} \tag{1.12}$$

For a pair of infinitely large parallel plates in vacuum:

$$\Phi_E = EA = \frac{\sigma A}{\epsilon_0} \tag{1.13}$$

$$\Rightarrow E = \frac{\sigma}{\epsilon_0}\,, \tag{1.14}$$

where $\sigma \equiv \frac{Q}{A}$ is the surface charge density.

1.4 Electric potential energy

$$W = \mathbf{F} \cdot \Delta\mathbf{x} \tag{1.15}$$

$$\Delta V = \mathbf{E} \cdot \Delta\mathbf{x} \tag{1.16}$$

For a point charge Q:

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{R} \quad \text{(when choosing } V = 0 \text{ at } R \to \infty) \tag{1.17}$$

1.5 Capacitance and dielectrics

$$C \equiv \frac{\Delta Q}{\Delta V} \tag{1.18}$$

For a parallel-plate Capacitor,

$$C = \epsilon\frac{A}{d} \tag{1.19}$$

The energy U stored in a capacitor with capacitance C is

$$U = \int_0^Q \mathrm{d}q\, V(q) \tag{1.20}$$

$$= \int_0^Q \mathrm{d}q\, \frac{q}{C} \tag{1.21}$$

$$= \frac{1}{2}\frac{q^2}{C}\bigg|_0^Q \tag{1.22}$$

$$= \frac{1}{2}\frac{Q^2}{C} \quad \left(= \frac{1}{2}QV = \frac{1}{2}CV^2\right) \tag{1.23}$$

1.6 Resistor

$$V = IR \tag{1.24}$$

$$R = \rho\frac{\ell}{A} \tag{1.25}$$